Chapter 5
Transportation Problems

- Transportation Problem
  - Network Representation
  - General LP Formulation
- Transportation Simplex Method:
  A Special-Purpose Solution Procedure

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Transportation, Assignment, and Transshipment Problems

- A <u>network model</u> is one which can be represented by a set of nodes, a set of arcs, and functions (e.g. costs, supplies, demands, etc.) associated with the arcs and/or nodes.
- Transportation, assignment, and transshipment problems of this chapter as well as the PERT/CPM problems (in another chapter) are all examples of network problems.

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Transportation, Assignment, and Transshipment Problems

- Each of the three models of this chapter can be formulated as linear programs and solved by general purpose linear programming codes.
- For each of the three models, if the right-hand side of the linear programming formulations are all integers, the optimal solution will be in terms of integer values for the decision variables.
- However, there are many computer packages (including *The Management Scientist*) that contain separate computer codes for these models which take advantage of their network structure.

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Transportation Problem

- The <u>transportation problem</u> seeks to minimize the total shipping costs of transporting goods from m origins (each with a supply s_i) to n destinations (each with a demand d_j), when the unit shipping cost from an origin, i, to a destination, j, is c_{ij}.
- The <u>network representation</u> for a transportation problem with two sources and three destinations is given on the next slide.

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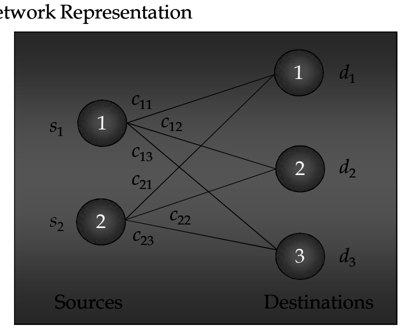
Transportation Problem

- Network Representation



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Transportation Problem

- LP Formulation

  The LP formulation in terms of the amounts shipped from the origins to the destinations, x_{ij}, can be written as:

$$\text{Min} \quad \sum_i \sum_j c_{ij} x_{ij}$$

$$\text{s.t.} \quad \sum_j x_{ij} \le s_i \quad \text{for each origin } i$$

$$\sum_i x_{ij} = d_j \quad \text{for each destination } j$$

$$x_{ij} \ge 0 \quad \text{for all } i \text{ and } j$$

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Transportation Problem

- LP Formulation Special Cases

  The following special-case modifications to the linear programming formulation can be made:
  - Minimum shipping guarantee from *i* to *j*:

    $x_{ij} \geq L_{ij}$
  - Maximum route capacity from *i* to *j*:

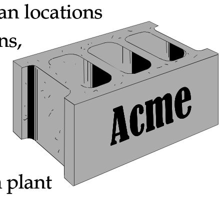
    $x_{ij} \leq L_{ij}$
  - Unacceptable route:

    Remove the corresponding decision variable.

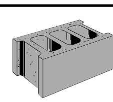
Example: Acme Block Co.

Acme Block Company has orders for 80 tons of concrete blocks at three suburban locations as follows: Northwood -- 25 tons, Westwood -- 45 tons, and Eastwood -- 10 tons. Acme has two plants, each of which can produce 50 tons per week. Delivery cost per ton from each plant to each suburban location is shown on the next slide.

How should end of week shipments be made to fill the above orders?



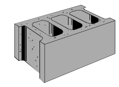
Example: Acme Block Co.



- Delivery Cost Per Ton

	Northwood	Westwood	Eastwood
Plant 1	24	30	40
Plant 2	30	40	42

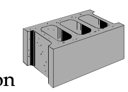
Example: Acme Block Co.



- Partial Spreadsheet Showing Problem Data

	A	B	C	D	E	F	G	H
1		LHS Coefficients						
2	Constraint	X11	X12	X13	X21	X22	X23	RHS
3	#1	1	1	1				50
4	#2				1	1	1	50
5	#3	1			1			25
6	#4		1			1		45
7	#5			1			1	10
8	Obj.Coefficients	24	30	40	30	40	42	30

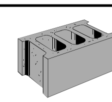
Example: Acme Block Co.



- Partial Spreadsheet Showing Optimal Solution

	A	B	C	D	E	F	G
10		X11	X12	X13	X21	X22	X23
11	Dec.Var.Values	5	45	0	20	0	10
12	Minimized Total Shipping Cost				2490		
13							
14	Constraints			LHS		RHS	
15	P1.Cap.			50	<=	50	
16	P2.Cap.			30	<=	50	
17	N.Dem.			25	=	25	
18	W.Dem.			45	=	45	
19	E.Dem.			10	=	10	

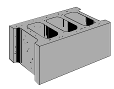
Example: Acme Block Co.



- Optimal Solution

From	To	Amount	Cost
Plant 1	Northwood	5	120
Plant 1	Westwood	45	1,350
Plant 2	Northwood	20	600
Plant 2	Eastwood	10	420
		Total Cost =	$2,490

Example: Acme Block Co.



- Partial Sensitivity Report (first half)

Adjustable Cells

Cell	Name	Final Value	Reduced Cost	Objective Coefficient	Allowable Increase	Allowable Decrease
C12	X11	5	0	24	4	4
D12	X12	45	0	30	4	1E+30
E12	X13	0	4	40	1E+30	4
F12	X21	20	0	30	4	4
G12	X22	0	4	40	1E+30	4
H12	X23	10.000	0.000	42	4	1E+30

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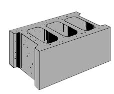
Example: Acme Block Co.



- Partial Sensitivity Report (second half)

Constraints

Cell	Name	Final Value	Shadow Price	Constraint R.H. Side	Allowable Increase	Allowable Decrease
E17	P2.Cap	30.0	0.0	50	1E+30	20
E18	N.Dem	25.0	30.0	25	20	20
E19	W.Dem	45.0	36.0	45	5	20
E20	E.Dem	10.0	42.0	10	20	10
E16	P1.Cap	50.0	-6.0	50	20	5

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Transportation Simplex Method

- To solve the transportation problem by its special purpose algorithm, the sum of the supplies at the origins must equal the sum of the demands at the destinations.
  - If the total supply is greater than the total demand, a dummy destination is added with demand equal to the excess supply, and shipping costs from all origins are zero.
  - Similarly, if total supply is less than total demand, a dummy origin is added.
- When solving a transportation problem by its special purpose algorithm, unacceptable shipping routes are given a cost of +*M* (a large number).

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Transportation Simplex Method

- A transportation tableau is given below. Each cell represents a shipping route (which is an arc on the network and a decision variable in the LP formulation), and the unit shipping costs are given in an upper right hand box in the cell.

	D1	D2	D3	Supply
S1	15	30	20	50
S2	30	40	35	30
Demand	25	45	10	

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Transportation Simplex Method

- The transportation problem is solved in two phases:
  - Phase I -- Finding an initial feasible solution
  - Phase II – Iterating to the optimal solution
- In Phase I, the Minimum-Cost Method can be used to establish an initial basic feasible solution without doing numerous iterations of the simplex method.
- In Phase II, the Stepping Stone Method, using the MODI method for evaluating the reduced costs may be used to move from the initial feasible solution to the optimal one.

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Transportation Simplex Method

- Phase I - Minimum-Cost Method
  - Step 1: Select the cell with the least cost. Assign to this cell the minimum of its remaining row supply or remaining column demand.
  - Step 2: Decrease the row and column availabilities by this amount and remove from consideration all other cells in the row or column with zero availability/demand. (If both are simultaneously reduced to 0, assign an allocation of 0 to any other unoccupied cell in the row or column before deleting both.) GO TO STEP 1.

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Transportation Simplex Method

- Phase II - Stepping Stone Method
  - Step 1: For each unoccupied cell, calculate the reduced cost by the MODI method described below.
    Select the unoccupied cell with the most negative reduced cost. (For maximization problems select the unoccupied cell with the largest reduced cost.) If none, STOP.
  - Step 2: For this unoccupied cell generate a stepping stone path by forming a closed loop with this cell and occupied cells by drawing connecting alternating horizontal and vertical lines between them.
    Determine the minimum allocation where a subtraction is to be made along this path.

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Transportation Simplex Method

- Phase II - Stepping Stone Method (continued)
  - Step 3: Add this allocation to all cells where additions are to be made, and subtract this allocation to all cells where subtractions are to be made along the stepping stone path.
    (Note: An occupied cell on the stepping stone path now becomes 0 (unoccupied). If more than one cell becomes 0, make only one unoccupied; make the others occupied with 0's.)
    GO TO STEP 1.

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Transportation Simplex Method

- MODI Method (for obtaining reduced costs)
  Associate a number, u_i, with each row and v_j with each column.
  - Step 1: Set $u_1 = 0$.
  - Step 2: Calculate the remaining u_i's and v_j's by solving the relationship $c_{ij} = u_i + v_j$ for occupied cells.
  - Step 3: For unoccupied cells (i,j), the reduced cost = $c_{ij} - u_i - v_j$.

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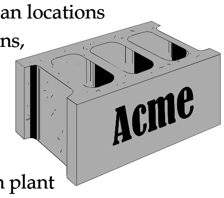
Example: Acme Block Co. (ABC)

Acme Block Company has orders for 80 tons of concrete blocks at three suburban locations as follows: Northwood -- 25 tons, Westwood -- 45 tons, and Eastwood -- 10 tons. Acme has two plants, each of which can produce 50 tons per week. Delivery cost per ton from each plant to each suburban location is shown on the next slide.



How should end of week shipments be made to fill the above orders?

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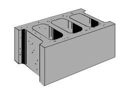
Example: ABC



- Delivery Cost Per Ton

	Northwood	Westwood	Eastwood
Plant 1	24	30	40
Plant 2	30	40	42

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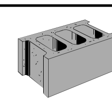
Example: ABC



- Initial Transportation Tableau
  Since total supply = 100 and total demand = 80, a dummy destination is created with demand of 20 and 0 unit costs.

	Northwood	Westwood	Eastwood	Dummy	Supply
Plant 1	24	30	40	0	50
Plant 2	30	40	42	0	50
Demand	25	45	10	20	

24

Example: ABC

- Least Cost Starting Procedure
  - Iteration 1: Tie for least cost (0), arbitrarily select x_{14}. Allocate 20. Reduce s_1 by 20 to 30 and delete the Dummy column.
  - Iteration 2: Of the remaining cells the least cost is 24 for x_{11}. Allocate 25. Reduce s_1 by 25 to 5 and eliminate the Northwood column.

continued ⟶

Example: ABC

- Least Cost Starting Procedure (continued)
  - Iteration 3: Of the remaining cells the least cost is 30 for x_{12}. Allocate 5. Reduce the Westwood column to 40 and eliminate the Plant 1 row.
  - Iteration 4: Since there is only one row with two cells left, make the final allocations of 40 and 10 to x_{22} and x_{23}, respectively.

Example: ABC

- Iteration 1
  - MODI Method
    1. Set $u_1 = 0$
    2. Since $u_1 + v_j = c_{1j}$ for occupied cells in row 1, then $v_1 = 24$, $v_2 = 30$, $v_4 = 0$.
    3. Since $u_i + v_2 = c_{i2}$ for occupied cells in column 2, then $u_2 + 30 = 40$, hence $u_2 = 10$.
    4. Since $u_2 + v_j = c_{2j}$ for occupied cells in row 2, then $10 + v_3 = 42$, hence $v_3 = 32$.

Example: ABC

- Iteration 1
  - MODI Method (continued)

    Calculate the reduced costs (circled numbers on the next slide) by $c_{ij} - u_i + v_j$.

Unoccupied Cell	Reduced Cost
(1,3)	40 - 0 - 32 = 8
(2,1)	30 - 24 -10 = -4
(2,4)	0 - 10 - 0 = -10

Example: ABC

- Iteration 1 Tableau

	Northwood	Westwood	Eastwood	Dummy	u_i
Plant 1	25 [24]	5 [30]	(+8) [40]	20 [0]	0
Plant 2	(-4) [30]	40 [40]	10 [42]	(-10) [0]	10
v_j	24	30	32	0	

Example: ABC

- Iteration 1
  - Stepping Stone Method

The stepping stone path for cell (2,4) is (2,4), (1,4), (1,2), (2,2). The allocations in the subtraction cells are 20 and 40, respectively. The minimum is 20, and hence reallocate 20 along this path. Thus for the next tableau:

x_{24} = 0 + 20 = 20 (0 is its current allocation)

x_{14} = 20 - 20 = 0 (blank for the next tableau)

x_{12} = 5 + 20 = 25

x_{22} = 40 - 20 = 20

The other occupied cells remain the same.

Example: ABC

- Iteration 2
  - MODI Method

    The reduced costs are found by calculating the u_i's and v_j's for this tableau.

    1. Set $u_1 = 0$.
    2. Since $u_1 + v_j = c_{ij}$ for occupied cells in row 1, then $v_1 = 24$, $v_2 = 30$.
    3. Since $u_i + v_2 = c_{i2}$ for occupied cells in column 2, then $u_2 + 30 = 40$, or $u_2 = 10$.
    4. Since $u_2 + v_j = c_{2j}$ for occupied cells in row 2, then $10 + v_3 = 42$ or $v_3 = 32$; and, $10 + v_4 = 0$ or $v_4 = -10$.

Example: ABC

- Iteration 2
  - MODI Method (continued)

    Calculate the reduced costs (circled numbers on the next slide) by c_{ij} - u_i + v_j.

Unoccupied Cell	Reduced Cost
(1,3)	40 - 0 - 32 = 8
(1,4)	0 - 0 - (-10) = 10
(2,1)	30 - 10 - 24 = -4

Example: ABC

- Iteration 2 Tableau

	Northwood	Westwood	Eastwood	Dummy	u_i
Plant 1	25 (cost 24)	25 (cost 30)	(+8) (cost 40)	(+10) (cost 0)	0
Plant 2	(-4) (cost 30)	20 (cost 40)	10 (cost 42)	20 (cost 0)	10
v_j	24	30	36	-6	

Example: ABC

- Iteration 2
  - Stepping Stone Method

    The most negative reduced cost is = -4 determined by x_{21}. The stepping stone path for this cell is (2,1),(1,1),(1,2),(2,2). The allocations in the subtraction cells are 25 and 20 respectively. Thus the new solution is obtained by reallocating 20 on the stepping stone path. Thus for the next tableau:

    x_{21} = 0 + 20 = 20 (0 is its current allocation)
    x_{11} = 25 - 20 = 5
    x_{12} = 25 + 20 = 45
    x_{22} = 20 - 20 = 0 (blank for the next tableau)
    The other occupied cells remain the same.

Example: ABC

- Iteration 3
  - MODI Method

    The reduced costs are found by calculating the u_i's and v_j's for this tableau.

    1. Set $u_1 = 0$
    2. Since $u_1 + v_j = c_{1j}$ for occupied cells in row 1, then $v_1 = 24$ and $v_2 = 30$.
    3. Since $u_i + v_1 = c_{i1}$ for occupied cells in column 2, then $u_2 + 24 = 30$ or $u_2 = 6$.
    4. Since $u_2 + v_j = c_{2j}$ for occupied cells in row 2, then $6 + v_3 = 42$ or $v_3 = 36$, and $6 + v_4 = 0$ or $v_4 = -6$.

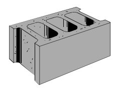
Example: ABC

- Iteration 3
  - MODI Method (continued)

    Calculate the reduced costs (circled numbers on the next slide) by c_{ij} - u_i + v_j.

Unoccupied Cell	Reduced Cost
(1,3)	40 - 0 - 36 = 4
(1,4)	0 - 0 - (-6) = 6
(2,2)	40 - 6 - 30 = 4

Example: ABC



- Iteration 3 Tableau

  Since all the reduced costs are non-negative, this is the optimal tableau.

	Northwood	Westwood	Eastwood	Dummy	u_i
Plant 1	5 (24)	45 (30)	+4 (40)	+6 (0)	0
Plant 2	20 (30)	+4 (40)	10 (42)	20 (0)	6
v_j	24	30	36	-6	

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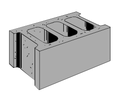
Example: ABC



- Optimal Solution

From	To	Amount	Cost
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Plant 2	Northwood	20	600
Plant 2	Eastwood	10	420
		Total Cost =	$2,490

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