Chapter 3
Linear Programming Applications

- The process of problem formulation
- Marketing and media applications
- Financial Applications
- Transportation Problem

The process of problem formulation

1. Provide a detailed verbal description of the problem
2. Determine the overall objective that appears to be relevant.
3. Determine the factors (constraints) that appear to restrict the attainment of the objective function.
4. Define the decision variables and state their units of measurement.
5. Using these decision variables, formulate an objective function.
6. Formulate a mathematical equations for each of the identified constraints.
7. Check the entire formulation to ensure linearity.

Marketing Applications

- One application of linear programming in marketing is media selection.
- LP can be used to help marketing managers allocate a fixed budget to various advertising media.
- The objective is to maximize reach, frequency, and quality of exposure.
- Restrictions on the allowable allocation usually arise during consideration of company policy, contract requirements, and media availability.
SMM Company recently developed a new instant salad machine, has $282,000 to spend on advertising. The product is to be initially test marketed in the Dallas area. The money is to be spent on a TV advertising blitz during one weekend (Friday, Saturday, and Sunday) in November.

The three options available are: daytime advertising, evening news advertising, and Sunday game-time advertising. A mixture of one-minute TV spots is desired.

### Media Selection

<table>
<thead>
<tr>
<th>Ad Type</th>
<th>Reached With Each Ad</th>
<th>Cost Per Ad</th>
</tr>
</thead>
<tbody>
<tr>
<td>Daytime</td>
<td>3,000</td>
<td>$5,000</td>
</tr>
<tr>
<td>Evening News</td>
<td>4,000</td>
<td>$7,000</td>
</tr>
<tr>
<td>Sunday Game</td>
<td>75,000</td>
<td>$100,000</td>
</tr>
</tbody>
</table>

SMM wants to take out at least one ad of each type (daytime, evening-news, and game-time). Further, there are only two game-time ad spots available. There are ten daytime spots and six evening news spots available daily. SMM wants to have at least 5 ads per day, but spend no more than $50,000 on Friday and no more than $75,000 on Saturday.

### Define the Decision Variables
- \( DFR \) = number of daytime ads on Friday
- \( DSA \) = number of daytime ads on Saturday
- \( DSU \) = number of daytime ads on Sunday
- \( EFR \) = number of evening ads on Friday
- \( ESA \) = number of evening ads on Saturday
- \( ESU \) = number of evening ads on Sunday
- \( GSU \) = number of game-time ads on Sunday
Media Selection

- Define the Objective Function
  Maximize the total audience reached:
  \[
  \text{Max} \quad \text{audience reached per ad of each type} \times \text{number of ads used of each type}
  \]
  \[
  \text{Max} \quad 3000\text{DFR} + 3000\text{DSA} + 3000\text{DSU} + 4000\text{EFR} + 4000\text{ESA} + 4000\text{ESU} + 75000\text{GSU}
  \]

- Define the Constraints
  - Take out at least one ad of each type:
    1. \( \text{DFR} + \text{DSA} + \text{DSU} \geq 1 \)
    2. \( \text{EFR} + \text{ESA} + \text{ESU} \geq 1 \)
    3. \( \text{GSU} \geq 1 \)
  - Ten daytime spots available:
    4. \( \text{DFR} \leq 10 \)
    5. \( \text{DSA} \leq 10 \)
    6. \( \text{DSU} \leq 10 \)
  - Six evening news spots available:
    7. \( \text{EFR} \leq 6 \)
    8. \( \text{ESA} \leq 6 \)
    9. \( \text{ESU} \leq 6 \)
  - Only two Sunday game-time ad spots available:
    10. \( \text{GSU} \leq 2 \)
  - At least 5 ads per day:
    11. \( \text{DFR} + \text{EFR} \geq 5 \)
    12. \( \text{DSA} + \text{ESA} \geq 5 \)
    13. \( \text{DSU} + \text{ESU} + \text{GSU} \geq 5 \)
  - Spend no more than $50,000 on Friday:
    14. \( 5000\text{DFR} + 7000\text{EFR} \leq 50000 \)
Media Selection

Define the Constraints (continued)

Spend no more than $75,000 on Saturday:
(15) \(5000 DSA + 7000 ESA \leq 75000\)

Spend no more than $282,000 in total:
(16) \(5000 DFR + 5000 DSA + 5000 DSU + 7000 EFR + 7000 ESA + 7000 ESU + 100000 GSU \leq 282000\)

Non-negativity:

\(DFR, DSA, DSU, EFR, ESA, ESU, GSU \geq 0\)

The Management Scientist Solution

Objective Function Value = 199000.000

<table>
<thead>
<tr>
<th>Variable</th>
<th>Value</th>
<th>Reduced Costs</th>
</tr>
</thead>
<tbody>
<tr>
<td>DFR</td>
<td>8.000</td>
<td>0.000</td>
</tr>
<tr>
<td>DSA</td>
<td>5.000</td>
<td>0.000</td>
</tr>
<tr>
<td>DSU</td>
<td>2.000</td>
<td>0.000</td>
</tr>
<tr>
<td>EFR</td>
<td>0.000</td>
<td>0.000</td>
</tr>
<tr>
<td>ESA</td>
<td>0.000</td>
<td>0.000</td>
</tr>
<tr>
<td>ESU</td>
<td>1.000</td>
<td>0.000</td>
</tr>
<tr>
<td>GSU</td>
<td>2.000</td>
<td>0.000</td>
</tr>
</tbody>
</table>

Solution Summary

Total new audience reached = 199,000

Number of daytime ads on Friday = 8
Number of daytime ads on Saturday = 5
Number of daytime ads on Sunday = 2
Number of evening ads on Friday = 0
Number of evening ads on Saturday = 0
Number of evening ads on Sunday = 1
Number of game-time ads on Sunday = 2
Financial Applications

- LP can be used in financial decision-making that involves capital budgeting, make-or-buy, asset allocation, portfolio selection, financial planning, and more.
- **Portfolio selection** problems involve choosing specific investments – for example, stocks and bonds – from a variety of investment alternatives.
- This type of problem is faced by managers of banks, mutual funds, and insurance companies.
- The objective function usually is maximization of expected return or minimization of risk.

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Portfolio Selection

Winslow Savings has $20 million available for investment. It wishes to invest over the next four months in such a way that it will maximize the total interest earned over the four month period as well as have at least $10 million available at the start of the fifth month for a high rise building venture in which it will be participating.

For the time being, Winslow wishes to invest only in 2-month government bonds (earning 2% over the 2-month period) and 3-month construction loans (earning 6% over the 3-month period). Each of these is available each month for investment. Funds not invested in these two investments are liquid and earn 3/4 of 1% per month when invested locally.
Portfolio Selection

Formulate a linear program that will help Winslow Savings determine how to invest over the next four months if at no time does it wish to have more than $8 million in either government bonds or construction loans.

**Define the Decision Variables**
- \( G_i \) = amount of new investment in government bonds in month \( i \) (for \( i = 1, 2, 3, 4 \))
- \( C_i \) = amount of new investment in construction loans in month \( i \) (for \( i = 1, 2, 3, 4 \))
- \( L_i \) = amount invested locally in month \( i \), (for \( i = 1, 2, 3, 4 \))

**Define the Objective Function**
Maximize total interest earned in the 4-month period:
Max (interest rate on investment) \( \times \) (amount invested)
\[
\text{Max } 0.02G_1 + 0.02G_2 + 0.02G_3 + 0.02G_4 \\
+ 0.06C_1 + 0.06C_2 + 0.06C_3 + 0.06C_4 \\
+ 0.0075L_1 + 0.0075L_2 + 0.0075L_3 + 0.0075L_4
\]
Portfolio Selection

- Define the Constraints

Month 1's total investment limited to $20 million:
(1) \( G_1 + C_1 + L_1 = 20,000,000 \)

Month 2's total investment limited to principle and interest invested locally in Month 1:
(2) \( G_2 + C_2 + L_2 = 1.0075L_1 \)
    or \( G_2 + C_2 - 1.0075L_1 + L_2 = 0 \)

- Define the Constraints (continued)

Month 3's total investment amount limited to principle and interest invested in government bonds in Month 1 and locally invested in Month 2:
(3) \( G_3 + C_3 + L_3 = 1.02G_2 + 1.0075L_2 \)
    or \(-1.02G_2 + G_3 + C_3 - 1.0075L_2 + L_3 = 0 \)

Month 4's total investment limited to principle and interest invested in construction loans in Month 1, government bonds in Month 2, and locally invested in Month 3:
(4) \( G_4 + C_4 + L_4 = 1.06C_3 + 1.02G_2 + 1.0075L_3 \)
    or \(-1.02G_2 + G_4 - 1.06C_3 + C_4 - 1.0075L_3 + L_4 = 0 \)

$10 million must be available at start of Month 5:
(5) \( 1.06C_2 + 1.02G_2 + 1.0075L_4 \geq 10,000,000 \)
Portfolio Selection

- Define the Constraints (continued)

No more than $8 million in government bonds at any time:

- \( G_1 \leq 8,000,000 \)
- \( G_1 + G_2 \leq 8,000,000 \)
- \( G_2 + G_3 \leq 8,000,000 \)
- \( G_3 + G_4 \leq 8,000,000 \)

Non-negativity:
\( G_i \geq 0 \text{ for } i = 1, 2, 3, 4 \)

Portfolio Selection

- Define the Constraints (continued)

No more than $8 million in construction loans at any time:

- \( C_1 \leq 8,000,000 \)
- \( C_1 + C_2 \leq 8,000,000 \)
- \( C_1 + C_2 + C_3 \leq 8,000,000 \)
- \( C_2 + C_3 + C_4 \leq 8,000,000 \)

Non-negativity:
\( C_i \geq 0 \text{ for } i = 1, 2, 3, 4 \)

Portfolio Selection

- *The Management Scientist Solution*

<table>
<thead>
<tr>
<th>Variable</th>
<th>Value</th>
<th>Reduced Costs</th>
</tr>
</thead>
<tbody>
<tr>
<td>G1</td>
<td>80000000.00</td>
<td>0.0000</td>
</tr>
<tr>
<td>G2</td>
<td>0.0000</td>
<td>0.0000</td>
</tr>
<tr>
<td>G3</td>
<td>5108613.92</td>
<td>0.0000</td>
</tr>
<tr>
<td>G4</td>
<td>2891386.07</td>
<td>0.0000</td>
</tr>
<tr>
<td>C1</td>
<td>8000000.00</td>
<td>0.0000</td>
</tr>
<tr>
<td>C2</td>
<td>0.0000</td>
<td>0.0453</td>
</tr>
<tr>
<td>C3</td>
<td>0.0000</td>
<td>0.0076</td>
</tr>
<tr>
<td>C4</td>
<td>8000000.00</td>
<td>0.0000</td>
</tr>
<tr>
<td>L1</td>
<td>4000000.00</td>
<td>0.0000</td>
</tr>
<tr>
<td>L2</td>
<td>4030000.00</td>
<td>0.0000</td>
</tr>
<tr>
<td>L3</td>
<td>7111611.07</td>
<td>0.0000</td>
</tr>
<tr>
<td>L4</td>
<td>4735562.08</td>
<td>0.0000</td>
</tr>
</tbody>
</table>

Objective Function Value = 1429213.7987
Transportation Problem

- The transportation problem seeks to minimize the total shipping costs of transporting goods from \( m \) origins (each with a supply \( s_i \)) to \( n \) destinations (each with a demand \( d_j \)), when the unit shipping cost from an origin, \( i \), to a destination, \( j \), is \( c_{ij} \).
- The network representation for a transportation problem with two sources and three destinations is given on the next slide.

Transportation Problem

- Network Representation

<table>
<thead>
<tr>
<th></th>
<th>1</th>
<th>2</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>Sources</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>2</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>3</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Destinations</td>
<td>d1</td>
<td>d2</td>
<td>d3</td>
</tr>
</tbody>
</table>

\( c_{11} \), \( c_{12} \), \( c_{13} \), \( c_{21} \), \( c_{22} \), \( c_{23} \)

Transportation Problem

- LP Formulation
  The LP formulation in terms of the amounts shipped from the origins to the destinations, \( x_{ij} \), can be written as:

\[
\begin{align*}
\text{Min} & \quad \sum_{ij} c_{ij}x_{ij} \\
\text{s.t.} & \quad \sum_{j} x_{ij} \leq s_i \text{ for each origin } i \\
& \quad \sum_{i} x_{ij} \geq d_j \text{ for each destination } j \\
& \quad x_{ij} \geq 0 \text{ for all } i \text{ and } j
\end{align*}
\]
Powerco has three electric power plants that supply the electric needs of four cities. The associated supply of each plant and demand of each city is given in the table 1. The cost of sending 1 million kwh of electricity from a plant to a city depends on the distance the electricity must travel. A transportation problem is specified by the supply, the demand, and the shipping costs. So the relevant data can be summarized in a transportation tableau. The transportation tableau implicitly expresses the supply and demand constraints and the shipping cost between each demand and supply point.

<table>
<thead>
<tr>
<th>From</th>
<th>To</th>
<th>City 1</th>
<th>City 2</th>
<th>City 3</th>
<th>City 4</th>
<th>Supply (Million kwh)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Plant 1</td>
<td>City 1</td>
<td>$8</td>
<td>$6</td>
<td>$10</td>
<td>$9</td>
<td>35</td>
</tr>
<tr>
<td>Plant 2</td>
<td>City 2</td>
<td>$9</td>
<td>$12</td>
<td>$13</td>
<td>$7</td>
<td>50</td>
</tr>
<tr>
<td>Plant 3</td>
<td>City 3</td>
<td>$14</td>
<td>$9</td>
<td>$16</td>
<td>$5</td>
<td>40</td>
</tr>
<tr>
<td>Demand (Million kwh)</td>
<td>City 4</td>
<td>45</td>
<td>20</td>
<td>30</td>
<td>30</td>
<td></td>
</tr>
</tbody>
</table>

Transportation Problem

1. Decision Variable:
   Since we have to determine how much electricity is sent from each plant to each city;
   \[ X_{ij} = \text{Amount of electricity produced at plant } i \text{ and sent to city } j \]
   \[ X_{14} = \text{Amount of electricity produced at plant 1 and sent to city 4} \]
Transportation Problem

2. Objective function
Since we want to minimize the total cost of shipping from plants to cities;

Minimize $Z = 8X_{11} + 6X_{12} + 10X_{13} + 9X_{14}$
+ $9X_{21} + 12X_{22} + 13X_{23} + 7X_{24}$
+ $14X_{31} + 9X_{32} + 16X_{33} + 5X_{34}$

Transportation Problem

3. Supply Constraints
Since each supply point has a limited production capacity;

$X_{11} + X_{12} + X_{13} + X_{14} \leq 35$
$X_{21} + X_{22} + X_{23} + X_{24} \leq 50$
$X_{31} + X_{32} + X_{33} + X_{34} \leq 40$

Transportation Problem

4. Demand Constraints
Since each supply point has a limited production capacity;

$X_{11} + X_{21} + X_{31} \geq 45$
$X_{12} + X_{22} + X_{32} \geq 20$
$X_{13} + X_{23} + X_{33} \geq 30$
$X_{14} + X_{24} + X_{34} \geq 30$
Transportation Problem

5. Sign Constraints
Since a negative amount of electricity can not be shipped all \( X_{ij} \)'s must be non negative;

\[ X_{ij} \geq 0 \quad (i=1,2,3; \ j=1,2,3,4) \]

LP Formulation of Powerco’s Problem

Min \( Z = 8X_{11} + 6X_{12} + 10X_{13} + 9X_{14} + 9X_{21} + 12X_{22} + 13X_{23} + 7X_{24} + 14X_{31} + 9X_{32} + 16X_{33} + 5X_{34} \)

S.T.: \[ X_{11} + X_{12} + X_{13} + X_{14} \leq 35 \quad \text{(Supply Constraints)} \]
\[ X_{21} + X_{22} + X_{23} + X_{24} \leq 50 \]
\[ X_{31} + X_{32} + X_{33} + X_{34} \leq 40 \]
\[ X_{11} + X_{12} + X_{13} \geq 45 \quad \text{(Demand Constraints)} \]
\[ X_{12} + X_{22} + X_{32} \geq 20 \]
\[ X_{13} + X_{23} + X_{33} \geq 30 \]
\[ X_{14} + X_{24} + X_{34} \geq 30 \]
\[ X_{ij} \geq 0 \quad (i=1,2,3; \ j=1,2,3,4) \]