# Linear Programming Formulations

For these problems you need to answer sensitivity analysis questions using excel. These questions appear in *italic fonts*. The excel files are available on the course website.

**Problem 1. Media Selection.**
The Westchester Chamber of Commerce periodically sponsors public service seminars and programs. Currently, promotional plans are under way for this year's program. Advertising alternatives include television, radio, and newspaper. Audience estimates, costs, and maximum media usage limitations are as shown.

| **Constraint** | **TV** | **Radio** | **Newspaper** |
|---|---|---|---|
| Audience per ad | 100,000 | 18,000 | 40,000 |
| Cost per ad | 2,000 | $ 300.00 | $600 |
| Maxuimum Media usage | 10 | 20 | 10 |

To ensure a balanced use of advertising media, radio advertisements must not exceed 50% of the total number of advertisements authorized. In addition, television should account for at least 10% of the total number of advertisements authorized.
a. If the promotional budget is limited to $18,200, how many commercial messages should be run on each medium to maximize total audience contact? What is the allocation of the budget among the three media, and what is the total audience reached?
*b. By how much would audience contact increase if an extra $100 were allocated to the advertising budget?*

**Problem 2. Investment and loans.**
The employee credit union at State University is planning the allocation of funds for the coming year. The credit union makes four types of loans to its members. In addition, the credit union invests in risk-free securities to stabilize income. The various revenue-producing investments together with annual rates of return are as follows.

| **Type of loan** | **Ann rate of return** |
|---|---|
| Automobile loans | 8 |
| Furniture loans | 10 |
| Other secured loans | 11 |
| Signature loans | 12 |
| Risk free securities | 9 |

The credit union will have $2,000,000 available for investment during the coming year. State laws and credit union policies impose the following restrictions on the composition of the loans and investments.

1. Risk-free securities may not exceed 30% of the total funds available for investment.
2. Signature loans may not exceed 10% of the funds invested in all loans (automobile, furniture, and other secured and signature loans).
3. Furniture loans plus other secured loans may not exceed the automobile loans.
4. Other secured loans plus signature loans may not exceed the funds invested in risk-free securities.

How should the $2,000,000 be allocated to each of the loan/investment alternatives to maximize total annual return? What is the projected total annual return?

**Problem 3. Make or Buy.**
Frandec Company manufactures, assembles, and rebuilds material hand equipment used in warehouses and distribution centers. One product, called a Liftmaster is assembled from four components: a frame, a motor, two supports, and a metal strap. Frandec is planning a production of 5000 Liftmasters next month. Frandec purchases the motors from an outside supplier, but the frames, supports, and straps may either be manufactured by the company or purchased from an outside supplier.
Manufacturing and purchase costs per unit are shown.

| Component | Manufacturing Costs | Purchase cost |
|---|---|---|
| **Frame** | $38.00 | $51.00 |
| **Support** | 11.50 | 15.00 |
| **Strap** | 6.50 | 7.50 |

Three departments are involved in the production of these components. The time (in min per unit) required to process each component in each department is given, along with available capacity (in hours) for the three departments.

| Component | Cutting Dept | Milling Dept | Shaping Dept |
|---|---|---|---|
| **Frame** | 3.5 | 2.2 | 3.1 |
| **Support** | 1.3 | 1.7 | 2.6 |
| **Strap** | 0.8 | - | 1.7 |
| **Capacity** | 350 | 420 | 680 |

a. Formulate and solve a linear programming model for this make-or-buy application. How many of each component should be manufactured and how many should be purchased?
b. What is the total cost of the manufacturing and purchasing plan?
c. How many hours of production time are used in each department?
*d. How much should Frandec be willing to pay for an additional hour of time in the shaping department?*
*e. Another manufacturer has offered to sell frames to Frandec for $45.00 each. Could Frandec improve its position by pursuing this opportunity? Why or why not?*

**Problem 4. Portfolio selection**
The Pfeiffer Company manages approximately $15 million for clients. For each client, Pfeiffer chooses a mix of three types of investments: a growth stock fund, an income fund, and a money market fund. Each client has different investment objectives and different tolerances for risk. To accommodate these differences, Pfeiffer places limits on the percentage of each portfolio that may be invested in the three funds and assigns a portfolio risk index to each client.

Here's how the system works for Dennis Hartmann, one of Pfeiffer's clients. Based on an evaluation of Hartmann's risk tolerance, Pfeiffer has assigned Hartmann's portfolio a risk index of 0.05. Furthermore, to maintain diversity, the fraction of Hartmann's portfolio invested in the growth and income funds must be at least 10% in each, and at least 20% must be invested in the money market fund.
The risk ratings for the growth, income, and money market funds are 0.10, 0.05, and 0.01, respectively. A portfolio risk index is computed as a weighted average of the risk ratings for the three funds where the weights are the fraction of the portfolio invested in each of the funds. Hartmann has given Pfeiffer $300,000 to manage. Pfeiffer is currently forecasting a yield of 20% on the growth fund, 10% on the income fund, and 6% on the money market fund.

a. Develop a linear programming model to select the best mix of investments for Hartmann's portfolio.
b. Solve the model you developed in part (a).
c. *How much may the yields on the three funds vary before Pfeiffer has to modify Hartmann's portfolio?*
d. *If Hartmann were more risk tolerant, how much of a yield increase could he expect? For instance, what if his portfolio risk index is increased to 0.06?*
e. *If Pfeiffer revised its yield estimate for the growth fund downward to 0.10, how would you recommend modifying Hartmann's portfolio?*

**Problem 5. Blending**

La Jolla Beverage Products is considering producing a wine cooler that would be a blend of a white wine, a rose wine, and fruit juice. To meet taste specifications, the wine cooler must consist of at least 50% white wine, at least 20% and no more than 30% rose, and 20% fruit juice. La Jolla purchases the wine from local wineries and the fruit juice from a processing plant in San Francisco. For the current production period, 10,000 gallons of white wine and 8000 gallons of rose wine can be purchased; there is no limit on the amount of fruit juice that can be ordered. The costs for the wine are $1.00 per gallon for the white and $1.50 per gallon for the rose; the fruit juice can be purchased for $0.50 per gallon. La Jolla Beverage Products can sell all the wine cooler it can produce for $2.50 per gallon.

a. Is the cost of the wine and fruit juice a sunk cost or a relevant cost in this situation? Explain.
b. Formulate a linear program to determine the blend of the three ingredients that will maximize the total profit contribution. Solve the linear program to determine the number of gallons of each ingredient La Jolla should purchase and the total profit contribution they will realize from this blend.
c. *If La Jolla could obtain additional amounts of the white wine, should they do so? If so, how much should they be willing to pay for each additional gallon and how many additional gallons would they want to purchase?*
d. *If La Jolla Beverage Products could obtain additional amounts of the rose wine, should they do so? If so, how much should they be willing to pay for each additional gallon and how many additional gallons would they want to purchase?*

**Problem 6. Transportation**

Tri-County Utilities, Inc., supplies natural gas to customers in a three-county area. The company purchases natural gas from two companies: Southern Gas and Northwest Gas. Demand forecasts for the coming winter season are Hamilton County, 400 units; Butler County, 200 units; and Clermont County, 300 units. Contracts to provide the following quantities have been written: Southern Gas, 500 units; and Northwest Gas, 400 units. Distribution costs for the counties vary, depending upon the location of the suppliers. The distribution costs per unit (in thousands of dollars) are as follows:

| **From/To** | **Hamilton** | **Butler** | **Clermont** |
|---|---|---|---|
| **Southern Gas** | 10 | 20 | 15 |
| **Northwest Gas** | 12 | 15 | 18 |

a. Develop a linear programming model that can be used to determine the plan that will minimize total distribution costs.
b. Describe the distribution plan and show the total distribution cost.

## Problem Answers

**Problem 1:**

a. Let

- T = number of television spot advertisements
- R = number of radio advertisements
- N = number of newspaper advertisements

| | | | |
|---|---|---|---|
| Max | $100{,}000T + 18{,}000R + 40{,}000N$ | | |
| s.t. | $2{,}000T + 300R + 600N \leq 18{,}200$ Budget | | |
| | $T \leq 10$ | Max TV | |
| | $R \leq 20$ | Max Radio | |
| | $N \leq 10$ | Max News | |
| | $-0.5T + 0.5R - 0.5N \leq 0$ | | Max 50% Radio |
| | $0.9T - 0.1R - 0.1N \leq 0$ | | Min 10% TV |
| | $T, R, N, \geq 0$ | | |

Solution:
$T = 4$
$R = 14$
$N = 10$
Audience = 1,052,000.

Budget :
$T$ : \$8,000,
$R$ : \$4,200,
$N$ : \$6,000

b. The dual price for the budget constraint is 51.30. Thus, a \$100 increase in budget should provide an increase in audience coverage of approximately 5,130. The right-hand-side range for the budget constraint will show this interpretation is correct.

**Problem 2:**

- $x1$ = \$ automobile loans
- $x2$ = \$ furniture loans
- $x3$ = \$ other secured loans
- $x4$ = \$ signature loans
- $x5$ = \$ "risk free" securities

Max $0.08x1 + 0.10x2 + 0.11x3 + 0.12x4 + 0.09x5$

s.t.
$x5 \leq 600{,}000$ [1]
$x4 \leq 0.10(x1 + x2 + x3 + x4)$ or $-0.10x1 - 0.10x2 - 0.10x3 + 0.90x4 \leq 0$ [2]
$x2 + x3 \leq x1$ or $- x1 + x2 + x3 \leq 0$ [3]
$x3 + x4 \leq x5$ or $+ x3 + x4 - x5 \leq 0$ [4]
$x1 + x2 + x3 + x4 + x5 = 2{,}000{,}000$ [5]
$x1, x2, x3, x4, x5 \geq 0$

Optimal solution:
Automobile Loans ($x1$) = \$630,000
Furniture Loans ($x2$) = \$170,000
Other Secured Loans ($x3$) = \$460,000
Signature Loans ($x4$) = \$140,000

Risk Free Loans ($x5$) = \$600,000
Annual Return \$188,800 (9.44%)

**Problem 3:**
a. Let
- FM = number of frames manufactured
- FP = number of frames purchased
- SM = number of supports manufactured
- SP = number of supports purchased
- TM = number of straps manufactured
- TP = number of straps purchased

Min $38FM + 51FP + 11.5SM + 15SP + 6.5TM + 7.5TP$
s.t. $3.5FM + 1.3SM + 0.8TM \leq 21{,}000$
$2.2FM + 1.7SM \leq 25{,}200$
$3.1FM + 2.6SM + 1.7TM \leq 40{,}800$
$FM + FP \geq 5{,}000$
$SM + SP \geq 10{,}000$
$TM + TP \geq 5{,}000$
$FM, FP, SM, SP, TM, TP \geq 0.$

Solution:

| | Manufacture | Purchase |
|---|---|---|
| Frames | 5000 | 0 |
| Supports | 2692 | 7308 |
| Straps | 0 | 5000 |

b. Total Cost = \$368,076.91

c. Subtract values of slack variables from minutes available to determine minutes used. Divide by 60 to determine hours of production time used.

| Constraint | | |
|---|---|---|
| 1 | Cutting: | Slack = 0 (350 hours used) |
| 2 | Milling: | (25200 - 9623) / 60 = 259.62 hours |
| 3 | Shaping: | (40800 - 18300) / 60 = 375 hours |

d. Nothing, there are already more hours available than are being used.

e. Yes. The current purchase price is \$51.00 and the reduced cost of 3.577 indicates that for a purchase price below \$47.423 the solution may improve. Resolving with the coefficient of FP =45 shows that 2714 frames should be purchased.

**Problem 4:**

a. Let $G$ = amount invested in growth stock fund
$S$ = amount invested in income stock fund
$M$ = amount invested in money market fund

Max $0.20G + 0.10S + 0.06M$

s.t. $0.10G + 0.05S + 0.01M \leq (0.05)(300{,}000)$ Hartmann's max risk
$G \geq (0.10)(300{,}000)$ Growth fund min.
$S \geq (0.10)(300{,}000)$ Income fund min.
$M \geq (0.20)(300{,}000)$ Money market min.
$G + S + M \leq 300{,}000$ Funds available
$G, S, M \geq 0$

b. The solution to Hartmann's portfolio mix problem is given.
Objective Function Value = 36000.000

| Variable | Value Reduced | Costs |
|---|---|---|
| G | 120000.000 | 0.000 |
| S | 30000.000 | 0.000 |
| M | 150000.000 | 0.000 |

| Constraint | Slack/Surplus | Dual Prices |
|---|---|---|
| 1 | 0.000 | 1.556 |
| 2 | 90000.000 | 0.000 |
| 3 | 0.000 | -0.022 |
| 4 | 90000.000 | 0.000 |
| 5 | 0.000 | 0.044 |

c. These are given by the objective coefficient ranges. The portfolio above will be optimal as long as the yields remain in the following intervals:

Growth stock $0.15 \leq c1 \leq 0.60$
Income stock No Lower Limit $< c2 \leq 0.122$
Money Market $0.02 \leq c3 \leq 0.20$

d. The dual price for the first constraint provides this information. A change in the risk index from 0.05 to 0.06 would increase the constraint RHS by 3000 (from 15,000 to 18,000). This is within the right-hand-side range, so the dual price of 1.556 is applicable. The value of the optimal solution would increase by (3000)(1.556) = 4668.
Hartmann's yield with a risk index of 0.05 is 36,000 / 300,000 = 0.12
His yield with a risk index of 0.06 would be 40,668 / 300,000 = 0.1356

e. This change is outside the objective coefficient range so we must re-solve the problem.

**Problem 5:**

a. Relevant cost since LaJolla Beverage Products can purchase wine and fruit juice on an as – needed basis.

b. Let

- $W$ = gallons of white wine
- $R$ = gallons of rose wine
- $F$ = gallons of fruit juice

Max $1.5W + 1R + 2F$

s.t. $0.5W - 0.5R - 0.5F \geq 0$ % white

$-0.2W + 0.8R - 0.2F \geq 0$ % rose minimum

$-0.3W + 0.7R - 0.3F \leq 0$ % rose maximum

$-0.2W - 0.2R + 0.8F = 0$ % fruit juice

$W \leq 10000$ Available white

$R \leq 8000$ Available rose

$W, R, F \geq 0$

Optimal Solution: $W$ = 10,000, $R$ = 6000, $F$ = 4000

Profit contribution = $29,000.

c. Since the cost of the wine is a relevant cost, the dual price of $2.90 is the maximum premium (over the normal price of $1.00) that LaJolla Beverage Products should be willing to pay to obtain one additional gallon of white wine. In other words, at a price of $3.90 = $2.90 + $1.00, the additional cost is exactly equal to the additional revenue.

d. No; only 6000 gallons of the rose are currently being used.

**Problem 6:**

a. Let $xij$ = amount shipped from supply node $i$ to demand node $j$.

Min $10x11 + 20x12 + 15x13 + 12x21 + 15x22 + 18x23$

s.t. $x11 + x12 + x13 \leq 500$

$x21 + x22 + x23 \leq 400$

$x11 + x21 \geq 400$

$x12 + x22 \geq 200$

$x13 + x23 \geq 300$

$xij \geq 0$ for all $i, j$

b. Optimal Solution

Amount Cost

Southern - Hamilton 200

Southern - Clermont 300

Northwest - Hamilton 200

Northwest - Butler 200

Total Cost $11,900